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Extension of Type-V and Type-IX Homogeneous Cosmologies

Aslam Abdullah[a,]* and Hishamuddin Zainuddin[b]

[a] *Faculty of Mechanical and Manufacturing Engineering*
UTHM
Batu Pahat, Malaysia
[b] *Institute for Mathematical Research*
UPM
Serdang, Malaysia

Abstract

Homogeneous cosmologies described by four-parameter transitive groups of isometries G_4 with an additional dimension of four-manifolds were obtained by extending Type-V and Type-IX cosmologies which involve isometry groups G_3 with transitive actions on three-manifolds. The group G_4 was specified via a translational symmetry over the extra dimension in order to extend a three-manifold metric which admits Bianchi Type-N (N=V, IX) to that of four-manifold. The extended metric of Types V and IX are solved explicitly.

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1. Introduction

The dynamics of the universe gives rise to a question of interest in cosmological study, which is the acceleration behavior of the universe that becomes a *fashionable* topic among the cosmologists. This phenomenon is what observational data indicate rather unexpectedly [1] and can be exhibited by introducing additional dimensions to the models of cosmology.

The history of additional dimensions can be traced back to the Kaluza-Klein theory in 1920s for which the dimensions are compactified in this scheme. Unlike ten spacetime dimensions [2] and thus six extra dimensions in string theory, Kaluza's unified theory of gravity and electromagnetism has only one extra dimension [3]. Another scheme known as brane worlds appeared in 1999 for which our universe is localized in a higher dimensional space.

The brane worlds are considered to be of five-dimensions [4], in general, where there is only a single additional dimension along which the branes are positioned. One time-dimension is *shared* by the bulk and the branes.

The solutions for the Killing equation in the case of four-manifold obtained in this study also consider one additional dimension. Spacelike sections of the brain are three-dimensional metrics embedded in those of four-dimensions. The manifolds of four dimensions as the spacelike sections of the bulk are therefore suggested in this study. These bulks should admit a transitive group of motions since they are considered homogeneous. This constraint ties-up spacelike sections of the brane universes with the concept of symmetry.

* Corresponding author. *E-mail address:*aslam@uthm.edu.my

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2. The Killing Equations

Given a manifold with metric g, the Killing equations describing its symmetries are expressed as

$$\sum \xi^{\alpha}\partial_{\alpha}g_{\mu\nu} + g_{\mu\alpha}\partial_{\nu}\xi^{\alpha} + g_{\nu\alpha}\partial_{\mu}\xi^{\alpha} = 0, \quad (1)$$

where ξ^{α} and $g_{\mu\nu}$ are the Killing vector and the manifold metric components, respectively. The Killing vector for a three-dimensional metric manifold is given by

$$X = \xi^2\partial_2 + \xi^3\partial_3 + \xi^4\partial_4.$$

We have to solve the Killing equations to find Type-V and Type-IX four-manifolds incorporating such symmetries in (1) (i.e. $\mathbf{B}^4_V$ and $\mathbf{B}^4_{IX}$).

By setting $\alpha = 2,3,4$, (1) becomes

$$\sum_{\alpha=2}^{4} \xi^{\alpha}\partial_{\alpha}g_{\mu\nu} + g_{\mu\alpha}\partial_{\nu}\xi^{\alpha} + g_{\nu\alpha}\partial_{\mu}\xi^{\alpha} = 0, \quad (2)$$

where (μ,ν) = (1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4).

It is worth to note that $g_{\mu\nu}$ in this system are the components of the four-manifold metric.

The maximum number of the group parameters r [5] that describes symmetries of the three-manifold is given by

$$r = \frac{n(n+1)}{2} = 6,$$

since $n = 3$ is the manifold's dimension. The complete group of isometries for three-manifold is generated by the maximum number of the Killing vectors X to which r corresponds. Since there are six different sets of the Killing vectors' components ξ^2,ξ^3,ξ^4 for every combination of (μ,ν), hence (2) consists of sixty equations, in general.

The Killing vectors' components (ξ^2,ξ^3,ξ^4) which generate Type-N are as follows†:

Type-V (0,1,0), (1,0,0), $(1,x^3,x^4)$.

Type-IX (0,1,0), $(\cos x^3, -\cot x^2 \sin x^3, \sin x^3/\sin x^2)$,
$(-\sin x^3, -\cot x^2 \cos x^3, \cos x^3/\sin x^2)$.

The number of non-trivial equations in (2) is reduced to thirty for each case, since there are only three Killing vectors X. We can obtain the solutions by substituting these components into (2), as shown in the following sections.

Assuming $\xi^1 \neq 0$, so that the complete isometry groups for a four-manifold B^4 is generated by

$$X = \xi^1\partial_1 + \xi^2\partial_2 + \xi^3\partial_3 + \xi^4\partial_4, \quad (3)$$

(1) becomes

$$\sum_{\alpha=1}^{4} \xi^{\alpha}\partial_{\alpha}g_{\mu\nu} + g_{\mu\alpha}\partial_{\nu}\xi^{\alpha} + g_{\nu\alpha}\partial_{\mu}\xi^{\alpha} = 0, \quad (4)$$

3. Four-Dimensional Type-V Manifold

Since g_{11}, g_{12}, and g_{22} being functions only of x^1, we are always free to set

$$g_{11} = 1, g_{12} = \theta^2(x^1), g_{22} = \psi^2(x^1),$$

where $\theta^2(x^1)$, $\psi^2(x^1)$ functions of x^1. Also,

$$g_{13} = g_{14} = g_{23} = g_{24} = e^{-x^2},$$

and

$$g_{33} = g_{34} = g_{44} = e^{-2x^2}.$$

The metric for B^4_{IV} is then represented in geodetic form as

$$\begin{aligned} ds^2 = {} & (dx^1)^2 + 2\theta^2(x^1)\, dx^1 dx^2 + \\ & 2e^{-x^2}(dx^1dx^3 + dx^1dx^4 + dx^2dx^3 + dx^2dx^4) + \\ & \psi^2(x^1)(dx^2)^2 + e^{-2x^2}\left((dx^3)^2 + 2dx^3dx^4 + (dx^4)^2\right). \end{aligned} \tag{5}$$

The components of the Killing vector (3) are given as

$$\begin{aligned} \xi^1 &= \xi^2 = 1, \\ \xi^3 &= x^3 + 1, \\ \xi^4 &= x^4 + 1, \end{aligned}$$

by choosing $X_4 = \partial_1$ to be the Killing vectors Lie algebra extra element. This element together with the rest generate four dimensional group of motion B^4, whose subgroup is Bianchi Type-V. Note that the additional group generated by ∂_1 is P_1. By substituting these components into (4) we have

$$\theta\theta' = 0, \psi\psi' = 0.$$

The last set of equations shows that

$$\theta^2(x^1) = a, \psi^2(x^1) = b,$$

where *a, b* are arbitrary constants. The metric (5) is then given by

$$\begin{aligned} ds^2 = {} & (dx^1)^2 + 2a\,dx^1dx^2 + \\ & 2e^{-x^2}\left(dx^1dx^3 + dx^1dx^4 + dx^2dx^3 + dx^2dx^4\right) + \\ & b(dx^2)^2 + e^{-2x^2}\left((dx^3)^2 + 2dx^3dx^4 + (dx^4)^2\right). \end{aligned}$$

4. Four-Dimensional Type-IX Manifold

We work out the extension in a way similar to that for the previous case (i.e. that of Type-V). The metric components are taken to be

$$
\begin{aligned}
g_{11} &= 1,\\
g_{12} &= \phi \cos x^4 + \psi \sin x^4,\\
g_{13} &= \sin x^2 (\phi \sin x^4 - \psi \cos x^4 + \gamma),\\
g_{14} &= \gamma \tan x^2,\\
g_{22} &= 2 \csc x^2 \varphi + 4 \cot x^2 \left(-\kappa \cos x^4 + \eta \sin x^4\right),\\
g_{23} &= -\cos x^2 \left(\eta \cos x^4 + \kappa \sin x^4\right) + \alpha \sin(2/\sqrt{7}\, x^4) +\\
&\quad \beta \cos(2/\sqrt{7}\, x^4),\\
g_{24} &= \eta \cos x^4 + \kappa \sin x^4,\\
g_{33} &= 4 \sin x^2 \cos x^2 \left(\eta \sin x^4 - \kappa \cos x^4\right) +\\
&\quad \frac{18}{7} \sin x^2 \left[-\frac{\sqrt{7}}{2} \alpha \cos\left(\frac{2}{\sqrt{7}} x^4\right) + \frac{\sqrt{7}}{2} \beta \sin\left(\frac{2}{\sqrt{7}} x^4\right) + \delta \right]\\
&\quad \lambda \sin x^2 \cos x^2 - \frac{4}{7} \delta \sin x^2,\\
g_{34} &= \lambda + \sin x^2 \left(\eta \sin x^4 - \kappa \cos x^4\right),\\
g_{44} &= \lambda \sec x^2,
\end{aligned}
$$

where ϕ, ψ, γ, κ, η, α, β, δ, and λ are functions of (x^1,x^2), and

$$
\varphi = \frac{\sqrt{7}}{2} \left[-\alpha \cos\left(\frac{2}{\sqrt{7}} x^4\right) + \beta \sin\left(\frac{2}{\sqrt{7}} x^4\right) + \frac{2}{\sqrt{7}} \delta \right].
$$

The metric for Bianchi Type-IX four-manifold B^4_{IX} is then represented in geodetic form as

$$
ds^2 = \sum_{\mu,\nu}^{1\ldots4} g_{\mu\nu} dx^\mu dx^\nu. \tag{6}
$$

By assuming translation symmetry over the extra variable x^1 and x^4, B^4 then admits a group of motions generated by

$$
\begin{aligned}
X_1 &= \partial_3,\\
X_2 &= \cos x^3 \partial_2 - \cot x^2 \sin x^3 \partial_3 + \sin x^3 / \sin x^2\, \partial_4,\\
X_3 &= -\sin x^3 \partial_2 - \cot x^2 \cos x^3 \partial_3 + \cos x^3 / \sin x^2\, \partial_4,\\
X_4 &= \partial_1,\\
X_5 &= \partial_4.
\end{aligned}
$$

The combinations of the components of these Killing vectors $(\xi^1,\xi^2,\xi^3,\xi^4)$ are given by (0,0,1,0), (0,cos x^3, -cot x^2 sin x^3, sin x^3/sin x^2), (0, - sin x^3, -cot x^2 cos x^3, cos x^3/sin x^2), (1,0,0,0), and (0,0,0,1).
Solving for the metric by substituting these components into (4) we have

$$
\begin{aligned}
g_{11} &= a,\\
g_{12} &= g_{23} = g_{24} = 0,\\
g_{13} &= b \cos x^2,\\
g_{14} &= b,\\
g_{22} &= 1,\\
g_{33} &= g_{22} \sin^2 x^2 + c \cos^2 x^2,\\
g_{43} &= c \cos x^2,\\
g_{44} &= c,
\end{aligned}
$$

where (a,b,c) constants.

The line element (6) is then given by

$$ds^2 = a dx^{1^2} + dx^{2^2} + dx^{3^2} + dx^{4^2} + 2\left[\cos x^2 (b dx^1 dx^3 + dx^3 dx^4) + b dx^1 dx^4\right],$$

where a, b constants.

5. Conclusion

The metric components of four-dimensional manifolds $\mathbf{B}_V^4$ and $\mathbf{B}_{IX}^4$ satisfy the Killing equations involving the components of the Killing vector X which generates the complete group of isometries for the corresponding three-manifolds. Therefore, both four-dimensional manifolds $\mathbf{B}_V^4$ and $\mathbf{B}_{IX}^4$ (re-)admit Bianchi Type-V and IX, respectively, and the (incomplete) group of isometries for Type-N (N=V, IX) three-manifolds is a subgroup of that for the resulted four-manifolds. This allows us to assume the extra translation Killing symmetry over the additional variable x^1 as shown in all two cases, and thus confirms the possibility of spacelike sections of the bulk in which the branes are positioned.

The framework of the rapidly developing ideas of brane worlds is a natural source of wormhole geometry [7]. Thus, a calculation of Lorentzian wormholes as smooth bridges between two manifolds of three dimensions B^3 embedded in, for instance, $\mathbf{B}_V^4$ and $\mathbf{B}_V^4$, may be a possible future research program.

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